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TRANSVERSAL HOLOMORPHIC STRUCTURES

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Among the most important global structures which one can introduce into a differentiable manifold are those obtained by requiring that the jacobians of the coordinate transformations belong to a given linear Lie group G. These structures are called (integrable) G-structures. An ordinary differentiable structure is a G-structure where G is the full linear group $GL(n, \mathbf{R})$. Complex analytic manifolds are obtained by $GL(n, \mathbf{C})$-structures. Another structure which has been intensively investigated is that of foliate structures obtained by the subgroups G of the real or complex general linear groups composed of transformations leaving invariant a linear subspace of euclidean space on which the linear group operates (see, for example, Reeb [18] or Kodaira-Spencer [14]).

Geometrically, a (real) foliation is a decomposition of a manifold M into disjoint connected sets $\{L_\alpha\}$ called the leaves of the foliation such that locally they are isomorphic to the family of horizontal lines $\mathbf{R}^n$ in $\mathbf{R}^{n+q}$.

Additional structure may be introduced in the foliation by controlling more carefully the way the leaves are attached. These are denominated transversal structures and were introduced by Haefliger [11].

In the present work we are interested in transversal holomorphic structures, that is, we assume that the leaves are glued together in a complex analytic manner. These are G-structures where $G = H^{n,q}$ is the subgroup of $GL(n + 2q, \mathbf{R})$ consisting of those matrices of the form

$$\begin{pmatrix} A & A' \\ 0 & A'' \end{pmatrix}$$

where $A \in GL(n, \mathbf{R})$, $A' \in M_{2q,n}(\mathbf{R})$, $A'' \in GL(q, \mathbf{C}) \hookrightarrow GL(2q, \mathbf{R})$.

An $H^{n,q}$-structure in a manifold M is then given by a covering of coordinate patches with coordinates $(x^1, \cdots, x^n; z^1, \cdots, z^q) = (x, z)$ such that the changes of coordinates are local diffeomorphisms of $\mathbf{R}^n \times \mathbf{C}^q$ of the form

$$f(x, z) = (f_1(x, z), f_2(z))$$

where f_2 is a holomorphic function of z alone.

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An $H^{0,q}$-structure is a complex analytic structure, and an $H^{n,q}$-manifold can be considered as a generalization of a complex manifold where the leaves take the role which was played by the points. This approach is specially appealing from a function theoretic point of view, and one can introduce a structure sheaf $\mathfrak{F}\mathcal{O}$ given by those local functions which are constant along the leaves and holomorphic in the transversal variables.

In §2 we prove an interesting finiteness property which states that for compact $H^{n,q}$-manifolds any sheaf locally isomorphic to $\mathfrak{F}\mathcal{O}^r$ has finite dimensional cohomology. This result is well known for complex manifolds and it informs us that these structures have similar finiteness properties as compact complex manifolds.

If M is an $H^{n,q}$-manifold, it is a natural question to ask about the deformations of M. That is, if M appears in a family $\{M_t\}$ of transversal holomorphic structures, what is the relation between M_t and M_0? In §3 we introduce the Kodaira-Spencer machinery for deformations, and as an application of the finiteness properties of the cohomology groups we are able to prove that the infinitesimal deformations form finite dimensional vector spaces. In §4 we analyze marked deformations of M obtained by fixing the topological type of the foliation.

In §5 we analyze the internal structure of the foliation by considering the leaf space, which is obtained by collapsing the leaves of the foliation to a point with the quotient topology. We study a special type of foliations, which we have called "partially Hausdorff" and are defined by the property that the union of the Hausdorff open sets of the leaf space is a Hausdorff nonempty open set (the regular set). We prove that the regular set has a natural structure of a normal complex analytic space.

As an application of the deformation theory we obtain a surprising finiteness property of the leaf space of a partially Hausdorff foliation of codimension 1, which states that every connected component of the regular set in the leaf space is a quasiprojective curve (a finite Riemann surface), and that except possibly for three types of curves there are only a finite number of components. This is a generalization and was originally motivated by Ahlfors' finiteness theorem for Kleinian groups.

Any holomorphic foliation of codimension 1 has an underlying $H^{2(n-1),1}$-structure obtained by forgetting all the holomorphic structure except the way the leaves are attached. If this foliation is partially Hausdorff, we then obtain that the regular set has the above stated finiteness properties. There does not seem to be a simple proof of this fact staying within the complex analytic category. Although the deformation of complex foliations are known to be finite dimensional we encounter nontrivial integrability conditions which obstruct obtaining any conclusion. But if we just remember the transversal holomorphic structure, we still have finiteness of infinitesimal deformations, but now we do not have any integrability conditions, so we can conclude the finiteness properties of the regular set. The $H^{n,q}$-category is the natural setting where this reasoning may be displayed.

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1. Transversal holomorphic structures

Let $(x^1, \cdots, x^n; z^1, \cdots, z^q) = (x, z)$ be coordinates in $\mathbf{R}^n \times \mathbf{C}^q$, where the first set are real variables and the second are complex variables. Consider the pseudogroup $H^{n,q}$ of local C^∞-diffeomorphism of $\mathbf{R}^n \times \mathbf{C}^q$ given by $f = (f_1, \cdots, f_n; g_1, \cdots, g_q)$ which satisfy:

(a) $\partial g_i/\partial x^j = 0$, $i = 1, \cdots, q; j = 1, \cdots, n$; i.e., g_i is a function of z alone.

(b) $\partial g_i/\partial \bar{z}^j = 0$, $i, j = 1, \cdots, q$; i.e., g_i is a holomorphic function of z.

Geometrically, the first condition means that we are leaving the horizontal lines invariant and the second that these lines are glued in a holomorphic manner. Clearly $H^{n,q}$ is a pseudogroup (see Kodaira-Morrow [13, p. 8] or Lawson [15, p. 2]).

Let M be a paracompact Hausdorff topological space. By a *system of $H^{n,q}$-coordinates* we mean a covering $\{U_i\}$ of M with topological homeomorphisms f^i: $U_i \to V_i$, V_i open in $\mathbf{R}^n \times \mathbf{C}^q$ such that $f^{ij} = f^i \circ (f^j)^{-1}$ is an element of $H^{n,q}$ whenever it is defined. Two systems are equivalent if their union is again a system of $H^{n,q}$-coordinates.

An $H^{n,q}$-structure in M is an equivalence class of systems of $H^{n,q}$-coordinates on M. A manifold M provided with an $H^{n,q}$-structure is said to have a *transversely holomorphic foliation of codimension* q, and M is called an $H^{n,q}$-manifold.

Let M be an $H^{n,q}$-manifold. We can introduce a second topology in M, callled the *leaf topology in* M, whose basis consists of sets of the form

$$\{p \in U | z^1(p) = \text{constant}, \cdots, z^q(p) = \text{constant}\}$$

where (x, z): $U \to \mathbf{R}^n \times \mathbf{C}^q$ is an arbitrary $H^{n,q}$-coordinate of M. In this topology the connected components of M, $\{L_\alpha\}$ are the *leaves* and M carries the structure of an uncountable n-dimensional manifold.

Each leaf L_α is a connected n-dimensional manifold embedded in M, but the embedding may not be proper, that is, the natural manifold topology on the leaf is not necessarily the one induced from M since the leaf may pass through a given chart infinitely often and accumulate on itself.

Example 1. If in the definition of $H^{n,q}$ we assume that $n = 2m$ and if we identify $\mathbf{R}^n$ with $\mathbf{C}^m$, we can consider the pseudogroup $H_c^{m,q}$ of local biholomorphic maps of $\mathbf{C}^m \times \mathbf{C}^q$ with complex variables (x, z) which satisfy condition (a); that is, if $f = (f_1, \cdots, f_m; g_1, \ldots, g_q)$, then $\partial g_i/\partial x^j = 0$, $i = 1, \cdots, q; j = 1, \cdots, n$. A manifold with an $H_c^{m,q}$-structure is in a natural way a complex manifold provided with a *holomorphic foliation of codimension* q. In this case, each leaf also possesses the structure of a complex manifold.

Example 2. Let X be an n-dimensional manifold, $\tilde{X} \to X$ its universal covering space, and $\pi_1(X)$ the fundamental group of X realized as covering transformations on $\tilde{X}$. Let F be a connected complex manifold of (complex) dimension q, and $\mathrm{Aut}_c(F)$ the group of biholomorphic automorphisms of F. Let

$$\rho\colon \pi_1(X) \to \mathrm{Aut}_c(F)$$

be any group homomorphism. $\pi_1(X)$ acts on $\tilde{X} \times F$ by

$$\gamma(p, t) = (\gamma(p), \rho(\gamma) \cdot t).$$

Since the action in $\tilde{X}$ consists of covering transformations, the action on $\tilde{X} \times F$ is properly discontinuous and without fixed points so the quotient space M has a manifold structure, and the projection map $\tilde{X} \times F \to M$ is a covering map. $\tilde{X} \times F$ has a natural structure of an $H^{n,q}$-manifold when we consider coordinates for $\tilde{X}$ and F separately $(x^1, \cdots, x^n; z^1, \cdots, z^q)$, and the leaves take the form $\{X \times t\}_{t \in F}$. The $H^{n,q}$-coordinates on $\tilde{X} \times F$ descend to M to induce an $H^{n,q}$-structure, where the leaves are now the images of $\{X \times t\}$. By projecting onto the first factor, M has a natural structure of a fiber bundle over X

$$M \to X$$

where the fibers are homeomorphic with F, and the leaves of the foliation are transversal to the fibers.

Two $H^{n,q}$-manifolds are *isomorphic* if there exists a diffeomorphism $\varphi\colon M \to M'$ such that in local $H^{n,q}$-coordinates $f^{-1}\varphi \circ g^{-1}$ belongs to $H^{n,q}$; i.e., φ preserves the leaves and is biholomorphic in the transversal variables.

The field of tangent spaces to the leaves of an $H^{n,q}$-manifold M form a vector bundle τ of the tangent bundle $T(M)$, called the *tangent bundle to the foliation*. The quotient bundle v is called the *normal bundle* of the foliation. We obtain an exact sequence of vector bundles over M:

$$0 \to \tau \to T(M) \to v \to 0. \qquad (1)$$

These bundles can be constructed explicitly as follows: Let (U_i, f^i) be a system of $H^{n,q}$-coordinates, $f^{ij}\colon f^i \circ (f^j)^{-1}\colon f^j(U_i \cap U_j) \to f^i(U_i \cap U_j)$ are the transition functions, local diffeomorphisms of $\mathbf{R}^n \times \mathbf{C}^q$ which can be written as

$$f^{ij} = \left(f_1^{ij}(x_j, z_j), f_2^{ij}(z_j)\right),$$

where $(x_j, z_j) = (x_j^1, \cdots, x_j^n; z_j^1, \cdots, z_j^q)$ are local coordinates for U_j. Letting D_1 be the derivative with respect to $(x^1, \cdots, x^n)$, and D_2 the derivative with respect to $(\operatorname{Re} z^1, \cdots, \operatorname{Re} z^q, \operatorname{Im} z^1, \ldots, \operatorname{Im} z^q)$, the transition functions of τ, $T(M)$ and v are given by the cocycles defined on $U_i \cap U_j$

$$(2) \qquad \tau_{ij} = D_1 f_1^{ij}, \quad T_{ij} = \begin{pmatrix} D_1 f_1^{ij} & D_2 f_1^{ij} \\ 0 & D_2 f_2^{ij} \end{pmatrix}, \quad v_{ij} = D_2 f_2^{ij}.$$

The transversal holomorphic structure allows us to define the *complex normal bundle* v_c obtained from the transition functions by considering only the complex derivatives of f_2^{ij}, with

$$v \otimes_{\mathbf{R}} \mathbf{C} = v_c \oplus \bar{v}_c.$$

(See Matsushima [17, p. 1] for more details.)

Lemma 1. *Let M be an $H^{n,q}$-manifold. Then the complex normal bundle v_c has a canonical representation by a cocycle whose functions are constant along the leaves, and v_c is obtained locally by pulling back the complex tangent bundle of the local submersions defining the foliation $\mathbf{R}^n \times \mathbf{C}^q \to \mathbf{C}^q$.*

Proof. Immediate from (2).

So we see that the normal bundle inherits a canonical complex structure; but this complex structure is special in the sense that if we express v in terms of the canonical cocycle (2), the endomorphism

$$I_i: v \otimes_{\mathbf{R}} \mathbf{C} \to v \otimes_{\mathbf{R}} \mathbf{C}$$

in U_i defining the complex structure is constant along the leaves.

We will now derive a differential geometric definition of an $H^{n,q}$-manifold.

Let M be a differentiable manifold of dimension m. A subbundle τ of $T(M)$ is *involutive* if for any two local vector fields X and Y with values in τ, their Lie bracket $[X, Y]$ is also a section of τ. By Frobenius Theorem (Chevalley [6, p. 89]) there exists a distinguished set of coordinate charts

$$\left(U_i, (x^1, \cdots, x^n; x^{n+1}, \cdots, x^m)\right)$$

in such a way that τ is the tangent space of

$$(3) \qquad x_{n+1} = \text{constant}, \cdots, \quad x_m = \text{constant},$$

and gives rise to a *foliation* in M. Let v be the normal bundle defined by (1). Then in terms of the distinguished coordinate charts the transition functions of the normal bundle are constant along the leaves, i.e., depend just on $x_{n+1}, \cdots, x_m$.

A *foliated almost complex structure* in v consists of an endomorphism

$$I: v \otimes_{\mathbf{R}} \mathbf{C} \to v \otimes_{\mathbf{R}} \mathbf{C}$$

with $I^2 = -\mathrm{id}$ such that in terms of the distinguished coordinates of v, $I_i = I_i(x_{n+1}, \cdots, x_m)$, i.e., is constant along the leaves.

Let I be a foliated almost complex structure in v, and let $(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m)$ be distinguished coordinates defined in U. Let $A(a_1, \cdots, a_n) = \{(a_1, \cdots, a_n; x_{n+1}, \cdots, x_n) \in U\}$ be a transversal to the foliation in U. By Lemma 1 the normal bundle restricted to $A(a_1, \cdots, a_n)$ is canonically identified with the tangent space to $A(a_1, \cdots, a_n)$. We say that I is *integrable* if for any distinguished coordinate and for every transversal $A(a_1, \cdots, a_n)$ the almost complex structure induced on $A(a_1, \cdots, a_n)$ is integrable; hence $A(a_1, \cdots, a_n)$ inherits a natural structure of a complex manifold.

Proposition 1. *Let M be a differentiable manifold. There is a one to one correspondence between $H^{n,q}$-structures on M and pairs (τ, I) such that*

(a) $\tau \hookrightarrow T(M)$ *is an involutive distribution of dimension n*; $n + 2q = \dim M$.

(b) *I is an integrable C^∞ foliated almost complex structure in the normal bundle.*

Proof. We have seen that any $H^{n,q}$-structure gives rise to (τ, I) with the stated properties. Conversely, suppose given (τ, I) as stated. τ gives rise to a set of distinguished coordinates as in (3). Choose coordinates of the form $A_1 \times A_2 \hookrightarrow \mathbf{R}^n \times \mathbf{R}^{2q}$. Then I induces an integrable almost complex structure on A_2, so perhaps after shrinking and a change of coordinates we may assume that $A_2 \hookrightarrow \mathbf{C}^q$ and the induced complex structure from $\mathbf{C}^q$ coincides with I. Let these be the new distinguished coordinates. Changes of coordinates will preserve the almost complex structures, so will be biholomorphic in the $(z_1, \cdots, z_q)$ coordinates, and hence we obtain an $H^{n,q}$-structure on M.

Suppose that (τ, I) and (τ', I') give rise to the same $H^{n,q}$-structure on M. Then $\tau = \tau'$ since the identity preserves the leaves; hence $v = v'$ and since $I = I'$ it is an isomorphism of almost complex structures. q.e.d.

In terms of the differentiable data two $H^{n,q}$-manifolds M and M' are isomorphic if and only if there exists a diffeomorphism $F: M \to M'$ such that

$$F_*(\tau) = \tau', F^*(I') = I.$$

Also note that for codimension one all foliated almost complex structures are integrable (see Lehto–Virtanen [16]).

2. Foliated sheaves

We fix throughout this section the notation $B = B' \times B'' \hookrightarrow \mathbf{R}^n \times \mathbf{C}^q$ where B' and B'' are the unit balls in $\mathbf{R}^n$ and $\mathbf{C}^q$ respectively.

Let M be an $H^{n,q}$-manifold, and (U_i, f^i) a covering by $H^{n,q}$-coordinate charts such that $f^i(U_i) = B \hookrightarrow \mathbf{R}^n \times \mathbf{C}^q$. Let d_1 be the differential operator defined in U_i obtained by taking the derivatives along the leaves; that is,

$$d_1\colon E(U_i, \textstyle\bigwedge^r \tau^*) \to E(U_i, \bigwedge^{r+1}\tau^*)$$

given by

$$d_1\Big(\sum_I f^I\, dx^I\Big) = \sum_I\Big(\sum_j \partial f^I/\partial x^j\, dx^j\Big) \wedge dx^I, \tag{4}$$

where $E(U_i, *)$ will always denote the C^∞ sections of the vector bundle $*$. Viewing B as a B''-parameter family of B'-disks, it is clear that d_1 is invariant under $H^{n,q}$ changes of coordinates, so gives rise to the operator

$$d_1\colon E(M, \textstyle\bigwedge^r \tau^*) \to E(M, \bigwedge^{r+1}\tau^*).$$

From the intrinsic nature of the operator, it induces a complex of sheaves

$$0 \to \mathcal{F}_M \to \mathcal{E}_M(C) \xrightarrow{d_1} \mathcal{E}_M(\textstyle\bigwedge^1\tau^*) \xrightarrow{d_1} \cdots \xrightarrow{d_1} \mathcal{E}_M(\bigwedge^n\tau^*) \to 0 \tag{5}$$

where $\mathcal{F}_M$ is the kernel of d_1 and consists of those local *functions which are constant along the leaves.*

Proposition 2. (5) *is a free resolution of sheaves, hence*

$$H^r(M, \mathcal{F}_M) = \frac{\{w \in E(M, \bigwedge^r\tau^*)/d_1(w) = 0\}}{d_1E(M, \bigwedge^{r-1}\tau^*)} = \frac{d_1\text{-closed}}{d_1\text{-exact}}.$$

Proof. $\mathcal{E}_M(\bigwedge^r\tau^*)$ are fine sheaves since we have C^∞ partitions of unity in M. To show that it is a free resolution it is enough to prove that locally the sheaves are exact, that is, that locally d_1-closed forms are d_1-exact. Choose local coordinates isomorphic to $A' \times A''$, where A' and A'' are open contractible open sets in $\mathbf{R}^n$ and $\mathbf{C}^q$ respectively, with coordinates (x, z). If w is a d_1-closed form we can view it as an A''-parameter family of closed forms in A', which can be integrated explicitly by

$$u(x, z) = \int_{(x_0,z)}^{(x,z)} \sum w^I\, dx^I. \tag{6}$$

So (5) is a free resolution. But then by Gunning-Rossi [**10**, p. 177] we know that the cohomology of $\mathcal{F}_M$ can be calculated from the complex of global sections of (5).

Corollary 1. *Let $A = A' \times A'' \hookrightarrow \mathbf{R}^n \times \mathbf{C}^q$ with A' and A'' open contractible sets. Then for any $s \geqslant 1$ we have*

$$H^0(A, \mathcal{F}_A) = E(A'', C)^s, \quad H^r(A, \mathcal{F}_A) = 0, \quad r > 0.$$

Proof. By the proposition we know how to calculate the cohomology of $\mathcal{F}_A$, but by (6) we have that the complex of global sections has trivial cohomology. This proves the corollary for $s = 1$. To prove it for $s > 1$ we use induction on

$$0 \to \mathcal{F}_A \to \mathcal{F}_A^s \to \mathcal{F}_A^{s-1} \to 0. \quad \text{q.e.d.}$$

If a vector bundle of rank r over M is given by a cocycle $\xi = (\xi_{ij})$ in such a way that the components $(\xi_\beta^\alpha)_{ij}$ are constant along the leaves, we say that ξ is a *foliated vector bundle* (i.e., $\xi \in H^1(M, GL(r, \mathcal{F}_M))$). In such a case, the sections of ξ which are constant along the leaves form a sheaf $\mathcal{F}(\xi)_M$ of locally free modules over $\mathcal{F}_M$. In particular, the complex normal bundle v_c is a foliated vector bundle of rank q by Lemma 1, and we also obtain the associated exterior algebra sheaves $\mathcal{F}_M(\bigwedge^r v_c^*)$.

From the $H^{n,q}$-structure on M we also obtain a $\bar\partial$-complex of sheaves

$$(7) \qquad 0 \to \mathcal{F}\mathcal{O}_M \to \mathcal{F}_M(C) \xrightarrow{\bar\partial} \mathcal{F}_M(\bar v_c^*) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{F}_M(\textstyle\bigwedge^q \bar v_c^*) \to 0,$$

$\bar\partial$ being defined in local coordinates $(x^1, \cdots, x^n; z^1, \cdots, z^q)$ by

$$\bar\partial\Big(\sum_I f^I(z)\, d\bar z^I\Big) = \sum_I (\partial f^I/\partial \bar z^1\, d\bar z^1 + \cdots + \partial f^I/\partial \bar z^q\, d\bar z^q) \wedge d\bar z^I,$$

where the first sheaf is the sheaf of *transversely holomorphic functions*. With the help of this sheaf, which will be called the *structure sheaf* of the $H^{n,q}$-manifold, we can give a sheaf theoretic definition of $H^{n,q}$-manifolds:

Proposition 3. *Let M be a differentiable manifold. There is a one to one correspondence between $H^{n,q}$-structures in M and subsheaves of the sheaf of C^∞ functions $\mathcal{E}_M(C)$ which are locally isomorphic to the structure sheaf $\mathcal{F}\mathcal{O}_{B' \times B''}$ where B' and B'' are the unit balls in $\mathbf{R}^n$ and $\mathbf{C}^q$ respectively.*

Proof. We have associated to every $H^{n,q}$-structure in M its structure sheaf which is locally isomorphic to $\mathcal{F}\mathcal{O}_{B' \times B''}$. By isomorphism we mean a diffeomorphism which is an isomorphism on the induced map of function sheaves.

Conversely, suppose given such a sheaf $\mathcal{G}$. Let U be an open set in M such that $\mathcal{G}_{|U} \cong \mathcal{F}\mathcal{O}_{B' \times B''}$. Then we pull back a foliation and an integrable foliated almost complex structure which glues correctly to any other trivialization. So by Proposition 1 we obtain an $H^{n,q}$-structure.

If two $H^{n,q}$-structures have the same sheaf, then they have the same leaves and the same transversal holomorphic structure, so they are the same. q.e.d.

If ξ is a vector bundle given by the cocycle (ξ_{ij}) in such a way that the entries of the matrix $(\xi_\beta^\alpha)_{ij}$ are transversely holomorphic functions, we say that ξ is an *h-foliated vector bundle* (i.e., $\xi \in H^1(M, GL(r, \mathcal{F}\mathcal{O}_M)))$. Given such a bundle we can construct the *sheaf of transversely holomorphic sections* $\mathcal{F}\mathcal{O}_M(\xi)$. Locally it is isomorphic to $\mathcal{F}\mathcal{O}_M^r$, and we glue it with the cocycle (ξ_{ij}). We will denominate such sheaves *h-foliated sheaves*. The objective of this section is to prove

Theorem 1. *Let $\mathcal{F}\mathcal{O}_M(\xi)$ be an h-foliated sheaf over the compact $H^{n,q}$-manifold M. Then all its cohomology groups $H^r(M, \mathcal{F}\mathcal{O}(\xi))$ are finite dimensional.*

We use Čech cohomology to compute cohomology. The proof runs parallel to a similar result for compact complex analytic spaces found in Gunning-Rossi [10, p 245] and can be divided into three parts:

1. We find suitable Leray coverings to compute cohomology.

2. We induce Frechet structures on the spaces of sections and prove that if $U \subset \bar{U} \subset\subset V$ then the restriction $H^0(V, \mathcal{F}\mathcal{O}(\xi)) \to H^0(U, \mathcal{F}\mathcal{O}(\xi))$ is a compact operator (i.e., we prove that the sheaves are 'Montel Sheaves').

3. We use Leray's theorem for cohomology of a covering together with a theorem of Schwartz to prove the finite dimensionality.

Proposition 4. *Let $A = A' \times A''$ be the product of two contractible open sets in $\mathbf{R}^n$ and $\mathbf{C}^q$ respectively, considered as an $H^{n,q}$-manifold. Then for every $s \geqslant 1$*

$$H^0(A, \mathcal{F}\mathcal{O}_A^s) = H^0(A'', \mathcal{O}_{A''})^s, \quad H^r(A, \mathcal{F}\mathcal{O}_A^s) = 0, r > 0.$$

Proof. We proceed by induction on s. Assume s = 1. Then the cohomology of the global sections of (7) is also trivial since we can explicitly integrate

$$u(x, z) = \int_{(x_0,z_0)}^{(x_0,z)} \sum w(z)^I \, d\bar{z}^I, \tag{8}$$

but these sheaves are not fine anymore. So instead of arguing as in Proposition 2, we consider the subsheaves of $\bar{\partial}$-closed forms $\mathcal{F}_A(\bigwedge^p \bar{v}_c^*)_c$ and start from the top

$$0 \to \mathcal{F}_A(\bigwedge^{q-1}\bar{v}_c^*)_c \to \mathcal{F}_A(\bigwedge^{q-1}\bar{v}_c^*) \xrightarrow{\bar{\partial}} \mathcal{F}_A(\bigwedge^q \bar{v}_c^*) \to 0.$$

By Proposition 2 the nonzero cohomology groups of the last two vanish, so $H^r(A, \mathcal{F}_A(\bigwedge^{q-1}\bar{v}_c^*)_c) = 0$ for $r \geqslant 2$. Using the long exact sequence we obtain

$$H^1(A, \mathcal{F}_A(\bigwedge^{q-1}\bar{v}_c^*)_c) = \frac{H^0(A, \mathcal{F}_A(\bigwedge^q \bar{v}_c^*))}{\bar{\partial} H^0(A, \mathcal{F}_A(\bigwedge^{q-1}\bar{v}_c^*))},$$

which is the cohomology of the global sections of the last term in (7), but by (8) it is zero. We proceed by descending induction on p until we get

$$0 \to \mathcal{F}\mathcal{O}_A \to \mathcal{F}_A(\mathbf{C}) \xrightarrow{\bar{\partial}} \mathcal{F}_A(\bigwedge^1 \bar{v}_c^*)_c \to 0.$$

By induction the nonzero cohomology of the last vanishes, by Proposition 2 the cohomology of the middle also vanishes and using again (7) and (8) we conclude that the nonzero cohomology of $\mathcal{F}\mathcal{O}_A$ also vanishes.

To finish the induction on s, we use the long exact sequence of

$$0 \to \mathcal{F}\mathcal{O}_A \to \mathcal{F}\mathcal{O}_A^s \to \mathcal{F}\mathcal{O}_A^{s-1} \to 0.$$

Lemma 2. *Let M be a compact $H^{n,q}$-manifold. We can always refine any covering to a finite cover $U = \{U_i\}$ such that each U_i is isomorphic to a product of balls $B' \times B'' \hookrightarrow \mathbf{R}^n \times \mathbf{C}^q$, and $U_{i_1} \cap \cdots \cap U_{i_m}$ is isomorphic to a product of open contractible sets $A' \times A''$ in $\mathbf{R}^n \times \mathbf{C}^q$. For these coverings we have*

$$H^r(M, \mathcal{F}\mathcal{O}_A(\xi)) = H^r(U, \mathcal{F}\mathcal{O}_A(\xi)) \quad r \geqslant 0$$

for any h-foliated vector bundle ξ.

Proof. Given any covering we construct a triangulation of M ' adapted to the foliation' (See Gromov [9] or Thurston [19]) such that every simplex is contained in an open set of the original covering. We subdivide this triangulation, and we take for U_i all stars of top dimensional simplices with their boundaries. Clearly this covering will satisfy the required properties. Restricting $\mathcal{F}\mathcal{O}_A(\xi)$ to U_i and using an isomorphism with $B' \times B''$ we obtain a holomorphic bundle on B'', but since all holomorphic bundles in the ball B'' are holomorphically trivial we conclude that $\mathcal{F}\mathcal{O}_A(\xi)|_{U_i} \cong \mathcal{F}\mathcal{O}_A^s$. Using Proposition 4 and Leray's theorem (Gunning-Rossi [10, p. 189]) we conclude that the cohomology of M can be computed with U. q.e.d.

Let U be an open set in M which is isomorphic to $B = B' \times B''$. Choose a trivialization of $\mathcal{F}\mathcal{O}(\xi)$ with $\mathcal{F}\mathcal{O}^s$. We induce a Frechet topology in $H^0(U, \mathcal{F}\mathcal{O}_M(\xi))$ by inducing on $H^0(U, \mathcal{F}\mathcal{O}_A^s)$ the Frechet topology of uniform convergence in compact subsets. This topology is independent of any of the choices. Let V be any open set, cover $V = \bigcup V_i$ by a locally finite covering each isomorphic to B, choose trivializations in each V_i of ξ, and induce the Frechet topologies. We induce on $H^0(V, \mathcal{F}\mathcal{O}_M(\xi))$ the Frechet topology induced by considering the pseudonorms obtained by restricting to each V_i. That this is well defined follows from similar arguments as in Gunning-Rossi [10, p. 237].

If $V \hookrightarrow U$, the restriction map $\rho_V^U\colon H^0(U, \mathcal{F}\mathcal{O}_M(\xi)) \to H^0(V, \mathcal{F}\mathcal{O}_M(\xi))$ is a continuous operator with respect to the Frechet topologies. Further, we have

Lemma 3. *If $V \subset \overline{V} \subset\subset U$, $\overline{V}$ compact, then the restriction map*

$$\rho_V^U\colon H^0(U, \mathcal{F}\mathcal{O}_M(\xi)) \to H^0(V, \mathcal{F}\mathcal{O}_M(\xi))$$

is a compact operator.

Proof. First assume that U is isomorphic to $B = B' \times B''$. Pick an open precompact set in B'' which contains the image of V into the second factor B''. Then

$$H^0(U, \mathcal{F}\mathcal{O}(\xi)) \to H^0(B' \times K, \mathcal{F}\mathcal{O}(\xi)) \to H^0(V, \mathcal{F}\mathcal{O}(\xi)),$$

and it is enough to show that $\rho_{B' \times K}^U$ is a compact operator. But this restriction map is isomorphic to

$$H^0(B'', \mathcal{O}_{B''}(\xi)) \to H^0(K, \mathcal{O}_{B''}(\xi))$$

which is a compact operator (This is Montel's theorem, which states that any bounded sequence of holomorphic functions has a uniform convergent subsequence in compact sets; see Gunning-Rossi [10, pp. 11 and 240]).

Let U be arbitrary. Choose a covering $U_1, \cdots, U_m$ of $\bar{V}$ such that each $\bar{U}_i$ is compact and contained in U. Let (f_n) be a bounded sequence, $\rho^U_{V_i}(f_n)$ is a bounded sequence for each i; by the above case we can choose a convergent subsequence for $U_1, \cdots, U_m$, hence a convergent subsequence for V.

Lemma 4. *Let M be a compact $H^{n,q}$-manifold, and $\mathfrak{F}\mathcal{O}_M(\xi)$ an h-foliated sheaf. Then $H^0(M, \mathfrak{F}\mathcal{O}_M(\xi))$ is a finite dimensional vector space.*

Proof. Since $M \subset \bar{M} \subset\subset M$, from Lemma 3 we obtain that the identity in $H^0(M, \mathfrak{F}\mathcal{O}_M(\xi))$ is a compact operator. Thus it is locally compact, and hence finite dimensional.

Proof of Theorem 1. Lemma 4 is the theorem for $p = 0$; so we may assume $p > 0$. Take a finite cover $U_1, \cdots, U_m$ of M such that $U_i \cong B' \times B''$. Let A' be an open contractible precompact set in B' such that $V_i = A' \times B''$ also cover M. The cochain groups

$$C^p(U, \mathfrak{F}\mathcal{O}_M(\xi)) = \bigoplus_{i_1, \cdots, i_p} H^0\big(U_{i_1} \cap \cdots \cap U_{i_p}, \mathfrak{F}\mathcal{O}_M(\xi)\big)$$

inherits the direct sum topology. By Lemma 3 the restriction map

$$\varphi^*: C^p(U, \mathfrak{F}\mathcal{O}_M(\xi)) \to C^p(V, \mathfrak{F}\mathcal{O}_M(\xi))$$

is a compact operator. The coboundary maps are continuous operators

$$\delta: C^p(U, \mathfrak{F}\mathcal{O}_M(\xi)) \to C^{p+1}(U, \mathfrak{F}\mathcal{O}_M(\xi)).$$

Hence

$$\varphi^*: Z^p(U, \mathfrak{F}\mathcal{O}_M(\xi)) \to Z^p(V, \mathfrak{F}\mathcal{O}_M(\xi))$$

is a compact operator. By Lemma 2 we have that

$$H^p(U, \mathfrak{F}\mathcal{O}_M(\xi)) = H^p(V, \mathfrak{F}\mathcal{O}_M(\xi)). \tag{9}$$

So consider

$$u: Z^p(U, \mathfrak{F}\mathcal{O}_M(\xi)) \oplus C^{p-1}(V, \mathfrak{F}\mathcal{O}_M(\xi)) \to Z^p(V, \mathfrak{F}\mathcal{O}(\xi)),$$
$$u(f, g) = \varphi^*(f) + \delta(g).$$

By (9) u is surjective. But then we can write $\delta = u - \varphi^* \oplus 0$ as a difference of a surjective map and a compact operator. By a theorem of Schwartz (Gunning-Rossi [10, p. 290]) u has closed image of finite codimension. Hence $H^p(M, \mathfrak{F}\mathcal{O}_M(\xi))$ is finite dimensional.

Remark. This proof works for singular foliations in the sense of Haefliger 11.

3. Differentiable families of deformations

Let X be a compact differentiable manifold of dimension $n = 2q$, and let U be an open set in the space of m real variables which contains 0. Let U be a spherical neighborhood of 0 in U. A *system of local* $H^{n,q}$*-coordinates* on $X \times U$ is a system $\{h^i\}$ of local differentiable coordinates of the form

$$h^i\colon (x, t) \to (h^i(x, t), t) \quad x \in X, t \in U$$

such that for each fixed t the maps

$$x \to h^i(x, t) = \left(h_1^i(x, t), \cdots, h_n^i(x, t), h_{n+1}^i(x, t), \cdots, h_{n+2q}^i(x, t)\right)$$

into $\mathbf{R}^n \times \mathbf{C}^q$ forms a system of local $H^{n,q}$-coordinates on X. A system of local $H^{n,q}$-coordinates on $X \times U$ defines a structure $M \overset{w}{\to} U$ of a *differentiable family of compact* $H^{n,q}$*-manifolds*, if each fiber $M_t = w^{-1}(t)$ of M is a compact $H^{n,q}$-manifold.

Let M be a compact $H^{n,q}$-manifold. By a *differentiable family of deformations of* M we mean a differentiable family $M \overset{w}{\to} U$ of compact $H^{n,q}$-manifolds such that $M = M_0 = w^{-1}(0)$. By a *differentiable family of small deformations* of M we mean the restriction $M|_{U_\varepsilon} = w^{-1}(U_\varepsilon)$ of a differentiable family $M \overset{w}{\to} U$ of deformations of $M = M_0$ to a sufficiently small neighborhood U_ε of 0.

At this point, following Kodaira and Spencer, it is useful to introduce the notion of jet forms for two reasons:

(a) Any differentiable family of deformations of M will be represented by a family $m(t)$ of jet forms on M depending differentiably on t.

(b) Jet forms will provide a free resolution of the sheaf of infinitesimal automorphisms of the $H^{n,q}$-structure.

Let $A = \bigoplus A^s$ be the graded algebra of real differentiable forms on M. By a *jet form of degree p on M* we mean a derivation of degree p on A, that is, a linear transformation of A satisfying

$$u(A^s) \hookrightarrow A^{s+p}, \quad u(\sigma \wedge \tau) = u\sigma \wedge \tau + (-1)^{ps}\sigma \wedge u(\tau), \quad \sigma \in A^s, \tau \in A.$$

One introduces a Lie algebra structure in the jet spaces by the formula

$$[u, v] = uv - (-1)^{rp}vu.$$

The exterior derivative is obviously a derivation of degree 1 on A. The *exterior derivative* of any jet formed is defined to be

$$Du = [d, u].$$

We denote by $\mathfrak{B}^p$ the sheaf over M of germs of differentiable vector p-forms (sections of $T(M) \otimes \bigwedge^p T(M)^*$), and by $\mathfrak{J}^p$ the sheaf over M of differentiable jet p-forms.

Theorem A (*Kodaira-Spencer* [14]). *Let M be a differentiable manifold of dimension m. Then the sequence of sheaves over M*

$$0 \to \mathcal{B}^0 \xrightarrow{i} \mathcal{J}^0 \xrightarrow{D} \mathcal{J}^1 \xrightarrow{D} \cdots \xrightarrow{D} \mathcal{J}^m \to 0$$

is exact, and $\mathcal{J}^p \approx \mathcal{B}^p \oplus \mathcal{B}^{p-1}$ is a canonical decomposition.

Let $(x^1, \cdots, x^n; z^1, \cdots, z^q)$ be $H^{n,q}$-coordinates for M. The complexified tangent bundle $T(M) \otimes_R C$ with these coordinates has two distinct complex basis:

1. Real basis: $\partial/\partial x^i$, $i = 1, \cdots, n + 2q$, where $z^j = x^{n+2j-1} + ix^{n+2j}$,

2. Semicomplex basis: $\partial/\partial x^i$, $i = 1, \cdots, n$, and $\partial/\partial z^j$, $\partial/\partial \bar{z}^j$, $j = 1, \cdots, q$. We modify slightly the notation and let $j = n + 1, \cdots, n + q$.

These bases extend to give local basis for the spaces of forms and vector forms.

By an *infinitesimal automorphism of the $H^{n,q}$-structure* we mean a differentiable real vector field θ such that in semicomplex basis

$$\theta = \sum_1^n \theta^i(x, z, \bar{z})\frac{\partial}{\partial x^i} + \sum_{n+1}^{q+n}\left[\theta^i(x, z, \bar{z})\frac{\partial}{\partial z^i} + \bar{\theta}^i(x, z, \bar{z})\frac{\partial}{\partial \bar{z}^i}\right]$$

satisfies the conditions

1. $\partial\theta^i/\partial x^j = 0$, $i = q + 1, \cdots, q + n; j = 1, \cdots, n$,
2. $\partial\theta^i/\partial \bar{z}^j = 0$, $i, j = q + 1, \cdots, q + n$.

The first condition reflects the foliation, and the second the transversal holomorphic structure. Denote by Θ the sheaf over M of germs of infinitesimal automorphisms.

Let u be any real p-form, and $(x^1, \cdots, x^n; z^{n+1}, \cdots, z^{n+q})$ local coordinates. Let β be the local $(p + 1)$-vector form described in real basis by

$$\beta = \sum_{i=1}^{n+2q} \beta^i\partial/\partial x^i, \quad \beta^i = (-1)^p u(dx^i).$$

We say that the *jet p-form belongs to the $H^{n,q}$-structure* if the components of β satisfy:

1. $\beta^i_{j,\cdots,k} = 0$; $i = n + 1, \cdots, n + 2q$; $j, \cdots, k = 1, \cdots, n$ for real basis,
2. $\beta^i_{j\cdots k} = 0$ for semicomplex basis for
   (i) $i = n + 1, \cdots, n + q; j, \cdots, k = \overline{n + 1}, \cdots, \overline{n + q}$, or
   (ii) $i = \overline{n + 1}, \cdots, \overline{n + q}; j, \cdots, k = n + 1, \cdots, n + q$.

Denote by Φ^p the subsheaf of $\mathcal{J}^p$ of all germs of differentiable jet p-forms belonging to the $H^{n,q}$-structure of M.

Consider a family $M \xrightarrow{w} U$ of deformations of M. Let $\{h^i\}$ be a system of local $H^{n,q}$-coordinates with transition functions

$$h^i(x, t) = g^{ik}(h^k(x, t), t)$$

and for any tangent vector

$$\partial/\partial t = \sum c_\gamma \partial/\partial t_\gamma$$

of U at the origin 0, define

$$\theta^{ik} = \sum_\alpha \theta_\alpha^{ik}(x)\partial/\partial x_i, \quad \theta_\alpha^{ik} = \partial g_\alpha^{ik}(x_k, t)/\partial t|_{t=0},$$

where $(x^1, \cdots, x^{n+2q})$ are real coordinates. The collection (θ^{ik}) of $H^{n,q}$-vector fields thus defined form a 1-cocycle on M, and its cohomology class will be called the infinitesimal deformation on M in the direction $\partial/\partial t$:

$$\rho_0(\partial/\partial t) \in H^1(M, \Theta) \tag{10}$$

Theorem B (*Kodaira-Spencer*). (1) $\Phi = \bigoplus \Phi^p$ *inherits a structure of a graded Lie algebra complex from* $\bigoplus \mathcal{J}^p$, *the sequence*

$$0 \to \Theta \xrightarrow{i} \Phi^0 \xrightarrow{D} \Phi^1 \xrightarrow{D} \cdots \xrightarrow{D} \Phi^{n+2q} \to 0$$

is exact and forms a free resolution of Θ.

(2) *Every differentiable family* $M \xrightarrow{w} U$ *of small deformations of* $M = w^{-1}(0)$ *determines a family* $\{m(t)/t \in U\}$ *of jet* 1-*forms* $m(t) \in H^0(M, \Phi^1)$ *satisfying*

$$[m(t), m(t)] = 0, \quad m(0) = d. \tag{11}$$

Conversely, given any family $\{m(t)/t \in U_1\}$ *of jet* 1-*forms* $m(t) \in H^0(M, \Phi^1)$ *on* M *depending differentiably on* $t \in U_1$ *and satisfying* (11) *there exist* $\varepsilon > 0$ *and a system of local* $H^{n,q}$-*coordinates* $\{h^i\}$ *on* $M \times U_\varepsilon$ *which defines a structure* $M \to U_\varepsilon$ *of a differentiable family of deformations of the* $H^{n,q}$-*manifold* M *which determines* $m(t)$ *in* U_ε.

(3) *The partial derivative* $(\partial m(t)/\partial t)_{t=0}$ *is an element of* $Z_D(\Phi^1)$ (*D-closed jets*) *and represents the infinitesimal deformation* $\rho_0(\partial/\partial t)$.

Remark. This theorem is essentially proved in Kodaira-Spencer [14]; the only difference consists in that the authors analyze separately the differentiable and complex analytic cases. So in order to prove it one would first apply their theorem for differentiable structures and then pose the complex analytic problem in the transversal directions. One then solves this one by applying their complex analytic theorem.

We now apply the finiteness theorem of §2 to prove that the infinitesimal deformations of compact $H^{n,q}$-manifolds are finite dimensional:

Theorem 2. *Let* M *be a compact* $H^{n,q}$-*manifold, and* Θ *the sheaf of germs of infinitesimal automorphisms. Then the cohomology groups*

$$H^p(M, \Theta), \quad p \geqslant 1,$$

are finite dimensional vector spaces.

Proof. Let

(12) $$0 \to \tau \to T(M) \xrightarrow{\pi} v \to 0$$

be the exact sequence of vector bundles which defines the foliation. Let v_c be the holomorphic normal bundle. We have an isomorphism between C^∞-sections of v_c and v given by

$$E(M, v_c) \to E(M, v), \quad f \to f + \bar{f}.$$

Θ consists of those sections of $T(M)$ such that when projected to the complex normal bundle via π and (12) are constant along the leaves and holomorphic in the transversal variables. Hence we get an exact sequence of sheaves

$$0 \to \mathcal{E}_M(\tau) \to \Theta \to \mathcal{F}\mathcal{O}_M(v_c) \to 0,$$

but $\mathcal{E}_M(\tau)$ is a fine sheaf since it consists of the C^∞-sections of τ; so from the long exact sequence and using the fact that $\mathcal{E}_M(\tau)$ is acyclic we obtain

(13) $$0 \to H^0(M, \mathcal{E}(\tau)) \to H^0(M, \Theta) \to H^0(M, \mathcal{F}\mathcal{O}(v_c)) \to 0,$$

(14) $$0 \to H^p(M, \Theta) \to H^p(M, \mathcal{F}\mathcal{O}(v_c)) \to 0, \quad p \geqslant 1.$$

By Theorem 1, $H^p(M, \mathcal{F}\mathcal{O}(v_c))$ is finite dimensional for $p \geqslant 0$; hence the theorem is proved.

Corollary 2. *Let M be a compact $H^{n,q}$-manifold. Then the infinitesimal-deformations of M form a finite dimensional vector space.*

Proof. The infinitesimal deformations of M are parametrized by $H^1(M, \Theta)$ which by the above theorem is finite dimensional.

Remark. The sequence (13) informs us that we can do the infinitesimal arguments in the normal bundle since there are no obstructions to lifting up to Θ. Note that $H^0(M, \Theta)$ is infinite dimensional since $H^0(M, \mathcal{E}(\tau))$ is. The isomorphisms (14) are the ones which justify the assertion that the deformations of M depend only on the transversal structure.

Also note that in case we were analyzing deformations of foliations (forgetting the holomorphic structure) we would get

$$H^p(M', \Theta') = H^p(M', \mathcal{F}(v')), \quad p \geqslant 1,$$

and the cohomology groups $H^p(M', \mathcal{F}(v'))$ could be infinite dimensional. Hence the finiteness theorem depends heavily in the holomorphic transversal structure.

4. Deformations with fixed topological type

In the deformation theory of complex manifolds as developed by Kodaira and Spencer there are two free resolutions of the sheaf of holomorphic vector fields, one given by the jets as in §3 and another given by the $\bar\partial$-operator. In this section we generalize the $\bar\partial$-sequence to an $H^{n,q}$-manifold, and obtain that this deformation theory consists of deformations with fixed topological type. The distinction arises from the fact that the $\bar{\partial}$-operator gives a resolution of the $H^{n,q}$-vector fields, but the sheaves are not fine anymore.

Let M be a compact manifold with a transversely holomorphic foliation. A *differentiable almost-complex family of marked deformations* $\mathcal{M} \xrightarrow{w} U$ of M is given by:

(i) An open neighborhood U of 0 in $\mathbf{R}^m$.

(ii) A trivializing diffeomorphism φ: $\mathcal{M} \to U \times M$.

(iii) A C^∞-parameter family of foliated almost-complex structures $\{I_t\}_{t\in U}$ in M (as in §1) such that I_0 is the usual foliated almost-complex structure of M.

The diffeomorphism φ allows us to define a foliation in each M_t by pulling back the foliation in M, and I_t defines a transversal almost-complex structure. $\mathcal{M} \xrightarrow{w} U$ is a *differentiable family of marked deformations* of M if I_t is an integrable almost-complex structure for each t. In this case each M_t has a canonical $H^{n,q}$-structure. By a *differentiable family of small marked deformations* of M we mean the restriction $\mathcal{M}|_{U_\varepsilon} = w^{-1}(U_\varepsilon)$ of a family $\mathcal{M} \xrightarrow{w} U$ of marked deformations of $M = M_0$ to a sufficiently small neighborhood U_ε of 0.

We define infinitesimal deformations as in (10).

Tensoring by v_c the exact sequence of sheaves (7) we obtain the following exact sequence of sheaves:

$$0 \to \mathcal{F}\Theta_M(v_c) \to \mathcal{F}_M(v_c) \xrightarrow{\bar{\partial}} \mathcal{F}_M(v_c \otimes \bar{v}_c^*) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{F}_M(v_c \otimes \textstyle\bigwedge^q \bar{v}_c^*) \to 0 \tag{15}$$

obtained by taking the usual $\bar{\partial}$-operator in the transversal variables on "transversal forms".

We can introduce a graded Lie algebra complex structure in (15) as follows. Let U be an open set in M which is isomorphic to $A' \times A'' \hookrightarrow \mathbf{R}^n \times \mathbf{C}^q$. Then we obtain an isomorphism of graded complexes

$$\begin{array}{ccccccccccc}
0 \to & \mathcal{F}\Theta_U(v_c) & \to & \mathcal{F}_U(v_c) & \xrightarrow{\bar{\partial}} & \mathcal{F}_U(v_c \otimes \bar{v}_c^*) & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & \mathcal{F}_U(v_c \otimes \bigwedge^q \bar{v}_c^*) & \to 0 \\
& \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\
0 \to & \Theta(TA'') & \to & \mathcal{E}(TA'') & \xrightarrow{\bar{\partial}} & \mathcal{E}(TA'' \otimes \bar{T}^*A'') & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & \mathcal{E}(TA'' \otimes \bigwedge^q \bar{T}^*A'') & \to 0
\end{array}$$

the lower sequence has a Lie algebra structure obtained by the Lie brackets of vector fields, so we pull up this structure. From the intrinsic nature of the Lie brackets of vector fields, this structure is independent of U and we obtain a graded Lie algebra structure on (15) as in Griffiths [8].

Theorem 3. *Every differentiable family of small marked deformations* $\mathcal{M} \xrightarrow{w} U$ *of the compact* $H^{n,q}$*-manifold* $M = M_0$ *determines a family of foliated vector* (0, 1)*-forms* $m(t) \in H^0(M, \mathfrak{F}(v_c \otimes \bar{v}_c^*))$ *satisfying*

$$\bar{\partial} m(t) - \tfrac{1}{2}[m(t), m(t)] = 0, \quad m(0) = 0. \tag{16}$$

Conversely, given any family $\{m(t) | t \in U_1\}$ of foliated vector (0, 1)-forms on M depending differentiably on $t \in U_1$ and satisfying (16), there exist $\varepsilon > 0$ and a system of local $H^{n,q}$-coordinates $\{h^i\}$ on $M \times U_\varepsilon$ which defines a structure $\mathcal{M} \to U_\varepsilon$ of a differentiable family of marked deformations of the $H^{n,q}$-manifold M which determines $m(t)$ in U_ε.

Proof. This is just a translation to our case of a classical result for complex manifolds. We follow Griffiths [8, p. 127].

By definition, $\mathcal{M}$ is determined by a differentiable family of foliated almost-complex structures $\{I_t\}_{t \in U}$. Each I_t is given by a family of "admissible frames" $e_t^\# = (e_{t1}, \cdots, e_{tq}; e_{t1}^*, \cdots, e_{tq}^*)$ where e_{ti} are elements of $v \otimes_{\mathbf{R}} C$ and e_{ti}^* is the complex conjugate of e_{ti}. We write $e_t^\# = (e_t, e_t^*)$. Given $e_t^\#$, the admissible frames are of the form $(Ae, \bar{A}e^*)$ where $A \in GL(q, C)$ (everything is constant along the leaves).

We let P_t and Q_t be the projections associated to I_t onto the vectors of type (0, 1) and (0, 1) respectively which I_t determines. Let $(x^1, \cdots, x^n; z^1, \cdots, z^q)$ be local $H^{n,q}$-coordinates on M. If I_t is close to I_0, then Q_0 will be nonsingular on Image(Q_t), and we may uniquely choose an I_t-admissible frame $e_t^\# = (e_t(z), e_t^*(z))$ such that $Q_0(e_t^*) = (\partial/\partial\bar{z}^1, \cdots, \partial/\partial\bar{z}^q)$. Then

$$e_{ti}^* = \partial/\partial\bar{z}^i - \sum_{j=1}^{q} m(t)_i^j \partial/\partial z^j, \quad e_{ti} = \partial/\partial z^i - \sum_{j=1}^{q} \overline{m}(t)_i^j \partial/\partial\bar{z}^j.$$

So $m(t) = \sum_{i,j} m(t)_i^j \partial/\partial z^i \otimes d\bar{z}^j$ is a tensor and defines an element in $H^0(M, \mathfrak{F}(v_c \otimes \bar{v}_c^*))$. Condition (16) is then seen to be equivalent to the integrability conditions in the Newlander-Nirenberg theorem (see Griffiths). Since the argument is reversible, the theorem follows. q.e.d.

Now that we know how to represent marked deformations, we would like to know which subspace of $H^1(M, \Theta)$ corresponds to marked infinitesimal deformations. For this, we use $\bar{\partial}$-closed forms and the sequence (15) to obtain a short exact sequence of sheaves

$$0 \to \mathfrak{F}\mathcal{O}_M(v_c) \to \mathfrak{F}_M(v_c) \xrightarrow{\bar{\partial}} \mathfrak{F}_M(v_c \otimes \bar{v}_c^*)_c \to 0.$$

The long exact sequence will give

$$\cdots \to H^0(M, \mathfrak{F}(v_c)) \xrightarrow{\bar{\partial}} H^0(M, \mathfrak{F}(v_c \otimes \bar{v}_c^*)_c) \xrightarrow{\delta} H^1(M, \mathfrak{F}\mathcal{O}(v_c)) \to \cdots,$$

and using (14) we obtain that the finite dimensional vector space

$$H_m = \frac{H^0(M, \mathcal{F}(v_c \otimes \bar{v}_c^*)_c)}{\bar{\partial} H^0(M, \mathcal{F}(v_c))} \hookrightarrow H^1(M, \Theta)$$

represents the infinitesimal marked deformations of M.

5. The leaf space

In this section we analyze the leaf space of a transversely holomorphic foliation obtained by collapsing the leaves of the foliation to a point with the quotient topology. We then analyze a special type of foliations, which we have called partially Hausdorff, and prove an interesting finiteness property when the foliation has complex codimension 1.

Let M be an $H^{n,q}$-manifold, and $\{L_\alpha\}$ the leaves of the underlying foliation. The *leaf space* C of M is defined to be the quotient topological space obtained by collapsing the leaves to a point; that is, the points of C are the leaves of the foliation, and we induce a topology on C by requiring that if π is the projection

$$\pi: M \to C,$$

U is open in C if and only if $\pi^{-1}(U)$ is open in M. The topology of C reflects the interrelations between the leaves.

Theorem 4. *Let M be an $H^{n,q}$-manifold, and U an open subset of the leaf space C of M which is Hausdorff. Then U has a natural structure of a normal analytic space.*

Proof. Let $t \in U$, and p a point in the leaf $\pi^{-1}(t)$. Let $(x^1, \cdots, x^n; z^1, \cdots, z^q)$ be coordinates around p contained in $\pi^{-1}(U)$ such that p has coordinates $(0, \cdots, 0; 0, \cdots, 0)$. Let $A = \{(0, \cdots, 0; z^1, \cdots, z^q) | \, |z^i| < \varepsilon\}$ be a small transversal to the foliation.

Every leaf L in $\pi^{-1}(U)$ is closed in U, hence $L \cap A$ is closed. On the other hand L intersects A at most in a countable number of times, so that actually $L \cap A$ is a discrete set of points (for otherwise it would be a perfect set, which has an uncountable number of elements).

The equivalence relation induced on A by π is then open (since it comes from a foliation), closed (since the quotient is Hausdorff, Bourbaki (5) p. 79) and discrete. By Holman [12, p. 338] we have that U has a natural structure of a normal complex anaytic space in a neighborhood of t; and since t is arbitrary and U is Hausdorff we have that U has a natural structure of a normal analytic space. q.e.d.

Let M be an $H^{n,q}$-manifold, and C its leaf space with projection $\pi: M \to C$. We define the *regular set* Ω of M to be the union of those leaves which have a

Hausdorff neighborhood in the leaf space. The complement $\Lambda = M - \Omega$ is called the *limit set*.

Definition. The foliation in the $H^{n,q}$-manifold M is *partially Hausdorff* if

1. the regular set Ω is nonempty,
2. $\pi(\Omega) \hookrightarrow C$ is a Hausdorff open set.

By definition $\pi(\Omega)$ is the union of Hausdorff open sets in C. Condition (2) then means that this union is actually Hausdorff. Condition (2) is nontrivial, as can be seen from the following example.

Example 3. Consider the group action

$$\mathbf{Z} \times (\mathbf{R}^1 \times P^2(\mathbf{C})) \to \mathbf{R}^1 \times P^2(\mathbf{C})$$

generated by

$$(g^1, g^2) = g(t; z^0 : z^1 : z^2) = \left(t + 1; \tfrac{1}{2}z^0 : 2z^1 : z^2\right).$$

This action is properly discontinuous and without fixed points, and the quotient M has a structure of an $H^{1,2}$-manifold as in Example 2 where the leaves are transverse to the $P^2(\mathbf{C})$-bundle structure over S^1:

$$M \to S^1.$$

Since each leaf intersects any fiber of the fibration $M \to S^1$, we have that the leaf space of M is homeomorphic to the orbit space on $P^2(C)/(g^2)$. Consider the open set U of P^2 given by affine coordinates (z^0, z^1), $z^2 = 1$. Then the group action in $U - (0, 0)$ is not properly discontinuous, although it is in $U - \{z^0 = 0\}$ and $U - \{z^1 = 0\}$. So $U - (0, 0)/(g^2)$ has the structure of a local analytic space but it is not Hausdorff. It seems likely that this happens also for codimension-1 foliations.

To motivate what follows, consider the example:

Example 4. A (geometrically) ruled surface S is a compact complex analytic surface with a holomorphic map to a complete complex curve B of genus g

$$\rho: S \to B \tag{17}$$

such that every fiber is isomorphic to a complex projective line.

The holomorphic foliations of S are described by immersions of complex line bundles into the holomorphic tangent bundle of S

$$\varepsilon \hookrightarrow T(S)$$

since in this case there are no integrability conditions. The ruling (17) determines a foliation, let

$$0 \to \tau \to T(S) \to v \to 0$$

be its defining sequence. We have

Theorem (*Gomez-Mont* [7]). (a) *If* $g \neq 1$ *and* $\xi \hookrightarrow T(S)$, *then* $\xi = \tau$ *as a subline bundle of* $T(S)$ *or* $\xi = v$ *as a complex analytic bundle.*

(b) *There exists a one to one correspondence between foliations arising from* v *and reductions of the bundle structure of* (17) *to a discrete structure group of* $PL(1, \mathbf{C})$.

This means that if $g > 1$ besides the ruling the holomorphic foliations are obtained by representations

$$\varphi: \pi_1(B) \to PL(1, \mathbf{R}) \oplus PL(1, \mathbf{C}),$$

where we act on $U \times P^1(\mathbf{C})$, U being the unit disk. The first component is universal representation $U \to B$, and the second is arbitrary. The leaves are the images of $\{U \times t\}_{t \in P^1}$ into S.

The leaf space can also be obtained as

$$P^1(C)/\varphi^2(\pi_1(B)),$$

and we see that there are two distinct types of foliations. If $\varphi^2(\pi_1(B))$ is not a Kleinian group, then every leaf of the foliation is dense. In case it is a Kleinian group, we have that the foliation is partially Hausdorff. In particular, Ahlfors' finiteness theorem (Ahlfors [1]) informs us that there are only a finite number of components in the regular set Ω and that each component has the structure of a quasiprojective curve (or a finite Riemann surface). We will now generalize this result to

Theorem 5. *Let* M *be a compact* $H^{n,1}$*-manifold with a locally Hausdorff foliation,* $\pi: M \to C$ *the map into the leaf space, and* Ω *the regular set. Then*

(a) *each connected component of* $\pi(\Omega)$ *is a quasiprojective curve,*

(b) *except possibly for* $P^1(\mathbf{C}) - \{0, 1$ *or* 2 *points*$\}$ *there are only a finite number of components of* $\pi(\Omega)$.

The proof is a direct generalization of Ahlfor's proof, and it is obtained by using marked infinitesimal deformations of M. The exceptions arise from the fact that $P^1(\mathbf{C}) - \{0, 1, 2 \text{ points}\}$ are rigid, so they cannot be detected by deformation theory. It seems very likely that only a finite number of these can occur.

Let M be an $H^{n,1}$-manifold, that is, M has a transversely holomorphic foliation of complex codimension 1. Denote by v_c its complex normal bundle. Using Theorem 3 we have that small deformations of the complex structure in v_c are parametrized by elements close to 0 in $H^0(M, \mathcal{F}(v_c \otimes \bar{v}_c^*))$. Given an element $\mu \in H^0(M, \mathcal{F}(v_c \otimes \bar{v}_c^*))$, $|\mu|$ is a real valued function so we can define its norm to be

$$\|\mu\| = \operatorname{Sup}\{|\mu(p)| \mid p \in M\}. \tag{18}$$

If μ has norm strictly less than one, we can pose in local coordinates $(x^1, \cdots, x^n; z)$ a Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu . \frac{\partial f}{\partial z}. \tag{19}$$

Beltrami equations always have local solutions defined up to conformal equivalence (Ahlfors [2]), so if we change coordinates

$$(x^1, \cdots, x^n; z) \to (x^1, \cdots, x^n; w) = (x^1, \cdots, x^n; f(z)),$$

we obtain from a covering $\{(U; x^1, \cdots, x^n; z)\}$ another $H^{n,1}$-structure on M given by $\{(U; x^1, \cdots, x^n; w)\}$.

More generally, let $L^\infty(M, \mathcal{F}(v_c \otimes \bar{v}_c^*))$ be the essentially bounded measurable sections of $v_c \otimes \bar{v}_c^*$ which are constant along the leaves. The elements of its unit ball $B^\infty(M, \mathcal{F}(v_c \otimes \bar{v}_c^*))$ will be denominated *Beltrami coefficients*. By solving in local coordinates the equation (19) (Lehto-Virtanen 16) we obtain a new $H^{n,1}$-structure on M which we denote by M_μ. Note that we might have to change the differentiable structure of M (to an equivalent one) since by definition M_μ has a C^∞-structure, and the identity might only be continuous since we are allowing arbitrary measurable coefficients.

We want to construct a deformation theory with fixed topological type in this wider context. The exact sequence which takes the place of (15) is

$$0 \to \mathcal{F}\mathcal{O}_M(v_c) \to \mathcal{F}\mathcal{C}(v_c) \xrightarrow{\bar\partial} L^\infty(M, \mathcal{F}(v_c \otimes \bar{v}_c^*)) \to 0 \tag{20}$$

where $F\mathcal{C}(v_c)$ is the sheaf of continuous sections constant along the leaves with distributional derivatives such that $\bar\partial f$ is in $L^\infty(M, \mathcal{F}(v_c \otimes \bar{v}_c^*))$.

Lemma 5. *The sequence of sheaves* (20) *is exact.*

Proof. The assertion is local. Since everything is constant along the leaves, we reduce it to a sequence in the complex plane, which is proved in Bers [4]. q.e.d.

Part of the associated long exact sequence is

$$\begin{aligned} &\to H^0(M, \mathcal{F}\mathcal{O}(v_c)) \to H^0(M, \mathcal{F}\mathcal{C}(v_c)) \xrightarrow{\bar\partial} L^\infty(M, \mathcal{F}(v_c \otimes \bar{v}_c^*)) \\ &\xrightarrow{\delta} H^1(M, \mathcal{F}\mathcal{O}(v_c)). \end{aligned} \tag{21}$$

By theorem 1 the two extremes are finite dimensional showing that $\bar\partial$ is an isomorphism except for finite dimensional spaces.

Assume now that the foliation in M is partially Hausdorff, and let $\Omega = \bigcup \Omega_i$ be its regular region, and $\pi(\Omega) = B = \bigcup B_i$ be the corresponding open set in the leaf space. We obtain an isomorphism

$$L^\infty(B, v_c \otimes \bar{v}_c^*) \to L^\infty(\Omega, \mathcal{F}(v_c \otimes \bar{v}_c^*)),$$

where the first are the usual Beltrami differentials in B.

The space of quadratic differentials on B has a norm given by integration

$$\int_B |\psi|$$

since $|\psi|$ is a 2-form. Let $L^1(B, T(B)^{-2})$ be the Banach space of integrable quadratic differentials with respect to this norm, and $A^1(B, T(B)^{-2})$ the closed subspace of the holomorphic integrable ones. Pulling these spaces to Ω, let $L^1(\Omega, \mathcal{F}(v_c^{-2}))$ and $A^1(\Omega, \mathcal{F}(v_c^{-2}))$ be the space of *integrable and integrable holomorphic sections of* v_c^{-2} *which are constant along the leaves.*

Lemma 6. $A^1(\Omega_i, \mathcal{F}(v_c^{-2}))$ *is finite dimensional if and only if* B_i *is of finite type* (*a quasiprojective curve*).

Proof. If B_i is a hyperbolic curve, this is [1, Theorem 1, p. 419]. If B_i is not hyperbolic, then it is always of finite type, and $A^1(B_i, T(B_i)^{-2})$ has dimension 0 or 1, so the lemma is proved.

Lemma 7. (a) *The pairing*

$$L^\infty(\Omega, \mathcal{F}(v_c \otimes \bar{v}_c^*)) \otimes L^1(\Omega, \mathcal{F}(v_c^{-2})) \to \mathbf{C}$$

given by

$$(\mu, \psi) \to \int_B \pi_*(\mu\psi) = \int_B \mu\psi$$

represents $L^\infty(\Omega, \mathcal{F}(v_c \otimes \bar{v}_c^*))$ *as the dual space of* $L^1(\Omega, \mathcal{F}(v_c^{-2}))$.

(b) *Under this duality, we have*

$$A^1(\Omega_i, \mathcal{F}(v_c^{-2}))^\perp = \bar{\partial} H^0(\Omega_i, \mathcal{F}\mathcal{C}(v_c)).$$

Proof. (a) Since the sections are constant along the leaves, we can consider them as sections in B. In B they become usual functions. Locally this corresponds to the usual duality between L^∞ and L^1, and globally they glue correctly.

(b) Since all the sections are constant along the leaves, we obtain an equivalent duality in B_i, and what we have to prove is that

$$A^1(B_i, T(B_i)^{-2})^\perp = \bar{\partial} H^0(B_i, \mathcal{C}(T(B_i))),$$

where $\mathcal{C}(T(B_i))$ is the sheaf of continuous vector fields in B_i with distributional derivatives such that $\bar{\partial}$ of them is essentially bounded.

If B_i is a hyperbolic curve, this is a classical statement (Ahlfors [2, p. 134]). If B_i is not hyperbolic, a case by case analysis gives the same result.

Proof of Theorem 5. (a) By Lemma 6 we have to prove that $A^1(B_i, T(B_i)^{-2})$ is finite dimensional, and by Lemma 7 this is equivalent to showing that $\bar{\partial} H^0(B_i, \mathcal{C}(T(B_i)))$ has finite codimension in $L^\infty(B_i, T(B_i) \otimes \overline{T(B_i)^*})$.

We obtain an inclusion

$$L^\infty(\Omega_i, \mathcal{F}(v_c \otimes \bar{v}_c^*)) \hookrightarrow L^\infty(M, \mathcal{F}(v_c \otimes \bar{v}_c^*))$$

by extending a section as 0 in $M - \Omega_i$; using (20) we have

$$\begin{array}{ccccc}
\cdots \to H^0(M, \mathcal{F}\mathcal{C}(v_c)) & \xrightarrow{\bar\partial} & L^\infty(M, \mathcal{F}(v_c \otimes \bar v_c^*)) & \xrightarrow{\delta} & H^1(M, \mathcal{F}\Theta(v_c)) \to \cdots \\
 & & \uparrow & & \\
\mathcal{F}\mathcal{C}(\Omega_i, v_c) & \xrightarrow{\bar\partial} & L^\infty(\Omega_i, \mathcal{F}(v_c \otimes \bar v_c^*)) & & \\
\| & & \| & & \\
\mathcal{C}(B_i, T(B_i)) & \xrightarrow{\bar\partial} & L^\infty(B_i, T(B_i) \otimes \bar T(B_i)^*). & &
\end{array}$$

If B_i is not compact, note that the last two expressions cannot be considered as the sections of a sheaf, since being essentially bounded is only a local property if the manifold is compact. Since $H^1(M, \mathcal{F}\Theta(v_c))$ is finite dimensional by Theorem 1, $\bar\partial H^0(M, \mathcal{F}\mathcal{C}(v_c))$ has finite codimension, so

$$\bar\partial H^0(M, \mathcal{F}\mathcal{C}(v_c)) \cap L^\infty(\Omega_i, \mathcal{F}(v_c \otimes \bar v_c^*)) \hookrightarrow L^\infty(\Omega_i, \mathcal{F}(v_c \otimes \bar v_c^*))$$

has finite codimension; which in turn implies that $\bar\partial\mathcal{C}(B_i, T(B_i))$ has finite codimenson in $L^\infty(B_i, T(B_i) \otimes \bar T(B_i)^*)$ which is what we wanted to prove.

(b) Let

$$(22) \quad D_i = \frac{L^\infty(\Omega_i, \mathcal{F}(v_c \otimes \bar v_c^*))}{\bar\partial H^0(M, \mathcal{F}\mathcal{C}(v_c)) \cap L^\infty(\Omega_i, \mathcal{F}(v_c \otimes \bar v_c^*))} \overset{\delta^*}{\hookrightarrow} H^1(M, \mathcal{F}\Theta(v_c)).$$

Let $\Omega_{i_1}, \cdots, \Omega_{i_m}$ be m distinct components. We have an inclusion (22) for each. If we prove that when we add (22) for $i_1, \cdots, i_m$, we still get an inclusion; this will imply that only for a finite number of i is

$$\bar\partial H^0(M, \mathcal{F}\mathcal{C}(v_c)) \not\supset L^\infty(\vartheta_i, \mathcal{F}(v_c \otimes \bar v_c^*))$$

since $H^1(M, \mathcal{F}\Theta(v_c))$ is finite dimensional. This will prove part (b).

Let

$$D_{i_1} \oplus D_{i_2} \oplus \cdots \oplus D_{i_m} \xrightarrow{\delta^*} H^1(M, \mathcal{F}\Theta(v_c)),$$

and suppose $\delta^*(\mu_{i_1} + \cdots + \mu_{i_m}) = 0$. Then this means that $\mu_{i_1} + \cdots + \mu_{i_m} = \delta\psi$ for some ψ; in particular, $\bar\partial\psi = \mu_{i_j}$ restricted to Ω_{i_j} since all the rest are zero there, so $\mu_{i_j} = 0$.

The problem reduces now to analyzing those curves which are rigid. They are known to be (Bers [3]) $P^1(\mathbf{C}) - \{0, 1, 2, 3 \text{ points}\}$. The first three are the exceptions in the statement of the theorem.

Let $B_i = P^1(C) - \{0, 1, \infty\}$. We follow Bers [4]. The idea consists in observing that although B_i does not have any integrable quadratic differentials, it will have higher order differentials. So if there were an infinite number of components isomorphic to B_i, we could get an infinite dimensional vector space, which contradicts Theorem 1. Formally we generalize the sequence (20) to

$$0 \to \mathcal{F}\mathcal{O}_M(v_c^r) \to \mathcal{F}\mathcal{C}_M(v_c^r) \xrightarrow{\bar\partial} L^\infty(M, v_c^r \otimes \bar{v}_c^*) \to 0.$$

Again we have that $H^1(M, \mathcal{F}\mathcal{O}(v_c^r))$ is finite dimensional, and repeating the argument as before we conclude that there are only a finite number of components which are isomorphic to B_i. For more details, see Bers [4].

Remarks. (1) The hypothesis are satisfied for a partially Hausdorff holomorhic foliation of codimension 1 in a compact complex manifold.

(2) We are using strongly the fact that any deformation of the leaf space can be lifted to a deformation of M. If we wanted to do this argument in the complex analytic category we would have to worry about giving "compatible" complex structures to the leaves, which becomes a nonlinear problem.

(3) For codimension $q \geqslant 2$, the integrability conditions obstruct this method to give any finiteness properties. It would be interesting to find out if the components of the regular set in the leaf space are analytic open sets in compact analytic space.

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